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6 SOME PROBLEMS OF FLOW, HEAT TRANSFER, AND DIFFUSION IN THE LAMINAR FLOW ALONG A FLAT PLATE

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SOME PROBLEMS OF FLOW, HEAT TRANSFER, AND DIFFUSION IN THE LAMINAR FLOW ALONG A FLAT PLATE

By

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I. Introduction

Motion of incompressible and gaseous media is represented by the Navier-Stoke differential equations. For two-dimensional flow and constant properties, these are, in Cartesian coordinates:

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = X - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$\rho \frac{\partial v}{\partial t} + \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = Y - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \qquad (1)$$

In addition to these, we have the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 , \qquad (2)$$

where u and v are the velocities in x and y directions, X and Y are the mass forces, p is the pressure, ρ the density, and μ the viscosity. To date, a solution of the non-linear differential equation (1) has not been found; not even if, as a simplification, we restrict ourselves to stationary properties which, from a mathematical point of view, will exclude all cases of turbulent flow. More responsive to a mathematical treatment are stationary flows at very small and at large Reynolds numbers.* In the first case, the convection terms ($u \frac{\partial u}{\partial x}$, $v \frac{\partial u}{\partial y}$, etc.) in Eq. (1) can be neglected; and in the second case, the effect of viscosity can be neglected as long as no flow separation takes place (potential flow). This is true except for a very thin layer directly on the surface of the boundary (boundary layer), in which the longitudinal velocity decreases with a very steep gradient to zero on the boundary, and in which the viscosity conditions may not be neglected. The flow in this boundary layer can also be treated mathematically as long as the layer remains laminar. Such boundary layers develop on all

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* Reynolds number is understood to be $\frac{Ud}{\nu}$, where U denotes a characteristic velocity, d a characteristic length, and ν the kinematic viscosity.

bodies subjected to a flow and are related to the flow resistance of the body. This originates from the forces acting on the body, which can be broken down into two components: Forces normal to the body (pressure forces); and those tangential to the body (shear stresses), due to the viscosity of the flowing medium. If no flow separation from the body takes place, the pressure forces can be calculated in a good approximation from the potential flow around the body; and the shear stresses, from the boundary layer flow.

A body for which, to begin with, nothing but shear stresses can be expected is a flat plate in longitudinal flow. For this, in 1904, Prandtl was first to show that, even at high Reynolds numbers, the drag could be calculated analytically from the Navier-Stoke equations. Here, the Reynolds numbers are assumed to be high, yet not too high, so that the flow remains laminar throughout ($Re = \frac{Ux}{\nu} < 300{,}000$, x= distance from the leading edge of the plate). Then, the thickness of the boundary layer normal to the plate (y - direction) is small as compared with its dimension in the direction of the plate (x - direction), so that, as was proven by Prandtl[1], we may limit ourselves in Eqs. (1) to the u-components; and further, $\frac{\partial^2 u}{\partial x^2}$ is small as to $\frac{\partial^2 u}{\partial y^2}$, so that it may be neglected; and we may write $\frac{dp}{dx} = 0$. Thus, for constant properties we obtain (introducing kinematic viscosity $\nu = \frac{\mu}{\rho}$):

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \qquad (3)$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (3a)$$

with the boundary conditions (U constant free-stream velocity in front of the plate)

$$\left.\begin{array}{lll} y = 0 & u = 0 & v = 0 \\ y = \infty & u = U & \end{array}\right\} . \qquad (4)$$

By introducing the non-dimensional distance from the boundary ξ and the stream function ψ,

$$\xi = \frac{y}{2}\sqrt{\frac{U}{\nu x}} \quad , \quad \psi = \sqrt{Ux\nu}\; f(\xi) \; . \qquad (5)$$

Equations (3) and (3a) can be reduced to common differential equations, where the continuity condition (3a) is satisfied, by writing $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$. Thus, we obtain

$$f''' + ff'' = 0 \quad . \tag{6}$$

A numerical solution of this differential equation, (6), was given by Blasius.[2] [Cf. Refs. 3, 4] Today, voluminous literature is available on boundary layers, even for those at pressure drops and pressure rises ($\frac{dp}{dx} \neq 0$).

If there is a temperature difference between the body and the flowing medium, a heat exchange between the two takes place, for which the heat conduction equation holds:

$$\rho\, c_p \frac{\partial T}{\partial t} + \rho\, c_p\, u \frac{\partial T}{\partial x} + \rho\, c_p\, v \frac{\partial T}{\partial y} = \lambda \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \quad , \tag{7}$$

where T denotes temperature, λ thermal conductivity, and c_p specific heat at constant pressure. To begin with, the heat produced by friction has not been considered in the differential equation (7). This is justified as long as the velocities encountered are not too high and the temperature differences are not too small. All conditions for the flow field are analogously valid for this case also; in particular, near the surface of the body a " thermal boundary layer" develops, in which the entire temperature drop between the body and the surrounding medium takes place.* If the corresponding flow field is already known, then the solution** of (7) for the flat plate at zero angle of attack can be reduced to a pure quadrature, as was shown by E. Polhausen:[5]

$$\theta = \frac{T - T_0}{T_1 - T_0} = \frac{K(\xi)}{K(\infty)} \; ; \quad K_{(\xi)} = \int_0^{\xi} e^{-Pr \int_0^{\xi} f d\xi} \, d\xi \quad , \tag{8}$$

where T_0 denotes the temperature at the wall, and T_1 that at the outer edge of the boundary layer***. Obviously, the boundary conditions $y = 0$, $T = T_0$, $y = \infty$, and $T = T_1$ are satisfied.

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* Therefore $\frac{\partial^2 T}{\partial x^2}$ may again be neglected with respect to $\frac{\partial^2 T}{\partial y^2}$; furthermore, we limit ourselves to stationary temperature fields ($\frac{\partial}{\partial t} = 0$).

** With the simplifying assumptions mentioned above.

*** Pr is the Prandtl number $\frac{\nu}{a}$, and f is defined by Eq. (5).

If, in the flow field, there are media of different kinds, the concentrations of which have local differences, then Eq. (7) is replaced by a similarly constructed diffusion equation in which the temperature T is replaced by the concentration c, and the thermal conductivity $a = \frac{\lambda}{\rho c_p}$ by the diffusion coefficient k.

In applications to practical problems of heat transfer and diffusion, difficulties arise due to the fact that the stipulation of constant properties is frequently not satisfied, since most properties are functions of temperature and then flow - and temperature fields are no longer independent of each other. A similar difficulty arises in the domain of heat transmission. On the basis of analogy considerations, we can represent, for constant properties, the results of heat transfer measurements in the form:

$$Nu = F(Re, Pr, \frac{\ell_1}{d}, \frac{\ell_2}{d} \ldots.) \quad , \qquad (9)$$

where Nu is the non-dimensional Nusselt number $\frac{\alpha d}{\lambda}$, $\frac{\ell_1}{d}$, $\frac{\ell_2}{d}$, etc., are the proportions determining the geometrical analogy conditions, and d is a characteristic dimension of length. The question arises whether the variability of the properties could not be taken into account by substituting into (9) the properties for a suitably selected temperature, so that, again, all measurements could be represented in a standard form.

In diffusion problems, the density may depend upon the concentration if the problem involves high concentrations and media of different densities. This corresponds to variable properties in temperature problems. However, another phenomenon may appear: At high concentrations, the normal velocity $v(0)$ no longer vanishes at the boundary, a fact pointed out previously by Nusselt.[6] If a liquid covers a wall and is made to evaporate by a gas flowing past this wall, then some liquid enters continously into the flow field and we obtain, therefore, $v(0) > 0$. If vapor condenses on a wall, or if a gas is bound to it by a chemical reaction, then $v(0) < 0$.

Presently there are two papers available on the laminar boundary layer on a flat plate in parallel flow with variable properties - one by L. Crocco[7] and the other by V. Kármán and Tsien.[8] In both papers, the differential equations are changed, by means of a transformation to new variables, into a form that is different from the common boundary layer equations. Crocco obtains two simultaneous, second degree, differential equations which he solves for a gas with the Prandtl number 0.725 (Air). Kármán and Tsien examine the special case of Pr = 1 and have to solve but one differential equation, since, in this case, the temperature field has a simple relation to the velocity field.

This study begins with an application of a method of solution (to which little reference is made in the literature) for the flow with constant properties over a flat plate. This method of solution was suggested originally by Piercy and Preston, employing Polhausen's formula (8). Futhermore, it is shown how the boundary layer flow with variable properties over the flat plate can be calculated simply and clearly by means of this solution method; and also, in diffusion problems, how the finite normal velocity at the boundary can be taken into account.

Let us limit ourselves to laminar boundary layers, since analytic treatment of the corresponding turbulent case is substantially more difficult and complicated. Laminar boundary layers appear on bodies in the vicinity of the stagnation point and extend from there over a distance that increases with decreasing Reynolds number, it being stipulated, of course, that no flow separation occurs. Such conditions present themselves particularly (1) in the flow of viscous fluids; (2) in cases in which the body in the flow is very small (e.g., particles in coal dust firing); and (3) in diffusion problems where, frequently, only low velocities appear.

Primarily, this study aims at fundamental information, while practical applications are possibly to be treated in a later report.

II. Solution of the Boundary Layer Equation for Variable Properties

Taking variable properties into account, the boundary layer equations for the velocity and temperature fields on the flat plate are:[5]

$$\rho\, u \frac{\partial u}{\partial x} + \rho\, v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y}\right) \tag{10}$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{10a}$$

$$\rho\, c_p\, u \frac{\partial T}{\partial x} + \rho\, c_p\, v \frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right) \tag{11}$$

The notations are the same as in (1), (2), and (7). Again, frictional heat is as yet neglected.

For constant properties, (10) and (11) can be reduced to a common differential equation (cf. introduction and Eqs. (3) through (6)) by the assumption that u and T are a function of but one (non-dimensional) coordinate, $\xi = \frac{y}{2}\sqrt{\frac{U}{\nu x}}$. Since the properties depend only upon temperature, the conclusion is obvious that the same simplification is possible for variable properties also. We write

$$\frac{u}{U} = \omega\,(\xi)\ ,\ \frac{T - T_0}{T_1 - T_0} = \theta\,(\xi),\ \ \xi = \frac{y}{2}\sqrt{\frac{U}{\nu_k x}} \tag{12}$$

where U is the velocity at the outside of the boundary layer, and T_0 and T_1 are the wall temperature and the temperature at the outer limit of the boundary layer, respectively. The magnitude ν_k in the non-dimensional ξ denotes the kinematic viscosity at the fixed temperature T_k, for which the wall temperature (k = 0), or the temperature at the outer limit of the boundary layer (k = 1), is a suitable selection. The boundary conditions for the flow and temperature fields are

$$\left.\begin{array}{l} y = 0,\ \xi = 0\ ,\ \omega = 0\ ,\ \ \theta = 0 \\ y = \infty,\ \xi = \infty,\ \omega = 1\ ,\ \ \theta = 1 \end{array}\right\} \tag{13}$$

Furthermore, let

$$\frac{\rho}{\rho_k} = \sigma(\xi)\ , \quad \frac{\mu}{\mu_k} = \phi(\xi)\ , \quad \frac{\lambda}{\lambda_k} = \chi(\xi)\ , \tag{14}$$

where the index k denotes the property at the temperature T_k .

The specific heat c_p is assumed to be constant, which is true for most gases and fluids. To begin with, we obtain from Eq. (10a)

$$\rho v = \sqrt{\frac{\nu_k}{Ux}}\left(\rho u \xi - \int_0^{\xi} \rho u \, d\xi\right)\ , \tag{15}$$

which leads from (10) and, after (14) and (15) have been introduced, to a form that is suitable for further calculations:

$$\frac{d}{d\xi}\left(\phi \frac{d\omega}{d\xi}\right) = -\left(\phi \frac{d\omega}{d\xi}\right)\frac{f}{\phi}\ ; \quad f = 2\int_0^{\xi} \sigma \omega \, d\xi\ . \tag{16}$$

From (16), the following expression can be derived for ω , if, temporarily, (16) is considered as a differential equation for $\phi \frac{d\omega}{d\xi}$; and, in addition, if f is regarded as a known function of ξ :

$$\omega = \frac{J(\xi)}{J(\infty)}\ ; \quad J_{(\xi)} = \int_0^{\xi} \frac{1}{\phi}\, e^{-\int_0^{\xi} \frac{f}{\phi} d\xi} \, d\xi\ , \tag{17}$$

where the constant of integration is given by the boundary condition (13). Similarly, we obtain for the non-dimensional temperature θ the expression

$$\theta = \frac{K(\xi)}{K(\infty)}\ ; \quad K_{(\xi)} = \int_0^{\xi} \frac{1}{\chi}\, e^{-Pr_k \int_0^{\xi} \frac{f}{\chi} d\xi} \, d\xi\ , \tag{18}$$

where $Pr_k = \frac{\nu_k}{\alpha_k} = \frac{\mu_k c_{pk}}{\lambda_k}$ denotes the Prandtl number with the properties at the temperature T_k.

For constant properties ($\phi = \sigma = \chi = 1$), the velocity and temperature fields are independent of each other, and we obtain from (18) the solution given by Polhausen[5] for the temperature field (cf. Eq. 8) which represents an integral equation for the velocity field for Pr = 1 ($\nu = \alpha$), as may be found by a comparison of (17) and (18). While, for a known velocity field, solution for the temperature field according to (8) or (18) is possible by means of simple quadrature, the calculation of the velocity field involves the difficulty that, in (17), the still unknown velocity appears on the right-hand side in the expression for f. In the method of solution given by Piercy and Preston, we proceed from an arbitrary approximation for ω which is then employed to calculate f according to (16), upon which $J(\xi)$ is calculated by means of (17). In this process it should be borne in mind that constant properties are stipulated, and, thus, that $\phi = \sigma = \chi = 1$. In this manner an improved value of ω is obtained which represents the initial value for a new calculation, etc. Figure 1 shows graphically the individual steps of the approximation. As initial solution $\omega^{(o)}$, the intentionally inaccurate approximation $\omega = 1$ over the entire boundary layer has been selected; the associated first approximation $\omega^{(1)}$ is given by the error integral. After the third approximation, the shear stress on the wall deviated but 4.5 percent from the exact value. Instead of continuing this process mechanically, the expected final solution was estimated from the course of the preceding approximations and then used as a basis for the subsequent step in the approximation. In this manner the solution ω was obtained with only 1/2-percent error in the shear stress.

For this method of solution, the improvement obtained by each approximation step can be estimated quantitatively: Eqs. (17) and (18) are identical for constant properties and Pr = 1. Let us assume that we dealt with an approximate solution $\omega^{(1)}$ of such a nature that the associated ξ coordinate differed by a constant factor γ from the ξ coordinate of the exact solution ω. Then, obviously, we have also $(f)^{(1)} = \gamma f$, and a comparison of (17) and (18) yields that the influence of the factor ν is equal to that of the quantity Pr

on the temperature field. Polhausen[5] found, on the basis of his numerical calculations, that the heat transfer coefficient is proportional to $\sqrt[3]{Pr}$; consequently, using the method mentioned above, the shear stress at the wall is subject to an error in every approximation step that is only about one-third of the error of the preceding one.

In the case of variable properties, these solution steps of a "mathematical" nature can be combined with steps of a "physical" nature:

Step 1

As the initial point, the known solutions for constant properties are assumed.

(a) For the velocity profile, the solution by Blasius.

(b) For the temperature field, the solution by E. Polhausen.

Step 2

(a) Calculation of the velocity profile according to (17), in which the temperature profile from step 1 (b) accounts for the temperature dependence of the properties.

(b) Calculation of the temperature field according to (18) by means of the velocity profile of step 2 (a); dependence of the properties upon temperature is taken into account in the same manner as in step 2 (a).

This procedure is repeated until the final solution is sufficiently exact. Generally, 3 to 4 repetitions will yield satisfactory results.

First, let us elaborate on the effect of the dependence of the properties on temperature. Of special interest in the velocity or temperature field are, chiefly, single values, such as the velocity gradient at the wall (for the calculation of the shear stress) or the heat transfer coefficient. The idea suggests itself to take into account the temperature dependence of the properties by substituting the properties at a suitably selected temperature into the "isothermal" formulas (i.e., the formulas for constant properties). If we select as reference temperature that at one of the two limits of the boundary layer, then, on the basis of physical aspects as well as on the basis of the formulas, it will be found that, within the boundary layer, an increase

in viscosity or density as compared with the values at the limits of the boundary layer will result in a drag increase as compared with the isothermal flow. Similarly, an increase in thermal conductivity and density results in increased thermal dissipation. However, the question, How great is the effect of the unconstancy of the properties? depends upon the ratio of the boundary layer thickness, of the temperature, and of the velocity field.*

Let us illustrate this by means of the following case, which has also some practical significance: The thermal boundary layer is assumed to be very small as compared with the flow boundary layer, a condition encountered at large Prandtl numbers (viscous fluids). In this case, then, the variation of the properties within the thermal boundary layer obviously may be neglected for the determination of the shear stress, which will be found to be the same as if the temperature at the outer edge of the boundary layer would extend to the wall. The same holds also for the velocity profile, with the exception of the small region within the thermal boundary layer in which the velocity profile is deformed in accordance with the viscosity variation. For the temperature profile, however, exactly this region is decisive. For the velocity gradient on the wall, we obtain, from the equality of the shear stress,

$$\left(\frac{\partial u}{\partial y}\right)_0 = \frac{\mu_1}{\mu_0}\left[\left(\frac{\partial u}{\partial y}\right)_0\right]_1 \quad \text{for } Pr \to \infty \,, \qquad (19)$$

where the index $_1$ denotes the "isothermal" flow with the properties at the temperature T_1. In this case, variability of density has no effect on the flow field. This can also be derived mathematically from Eqs. (17) and (18). The conditions for the temperature field are discussed in the following paragraph by means of two examples.

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* According to E. Polhausen, the ratio of the two is proportional to $\frac{1}{\sqrt[3]{Pr}}$.

III. Flow and Temperature Fields in Viscous Fluids

In accordance with the nature of the properties of viscous fluids, the velocity and temperature fields were calculated with the assumption that only viscosity varies with temperature according to the following formula:

$$\frac{\mu}{\mu_k} = \left(\frac{T_k + T_c}{T + T_c}\right)^b , \qquad (20)$$

where b and T_c are constants which are to be selected in such a manner that the temperature dependence is as well represented as possible. Let the index k have the value 0 or 1, depending on the selection of the non-dimensional ξ. Here, $b = 3$ was selected (e.g., viscous lubricating oil), and the two cases of a heated and cooled plate were calculated with $\frac{\mu_0}{\mu_1} = \frac{1}{8}$ and 8, respectively, and $Pr_0 = 12.5$ and 100. Thus, the two examples deal with the same fluid and involve equal temperature differences, since, in both cases, the respective Pr is formed with the properties of the wall temperature. If T_0 is selected as reference temperature, we obtain from (20)

$$\phi = \frac{\mu}{\mu_0} = \left(\frac{1}{\theta\left(\sqrt[3]{\frac{\mu_0}{\mu_1}} - 1\right) + 1}\right)^3 . \qquad (21)$$

The result of the calculations according to the iterative method of the preceding paragraph may be recognized in Figs. 2 and 3. In both graphs the non-dimensional distances from the wall, ξ_0 and ξ_1, formed with μ_0 and μ_1, are plotted on the abscissa at the scale 1: $\sqrt{8}$ (Fig. 2) and $\sqrt{8}$: 1 (Fig. 3), so that but one point on the abscissa corresponds to every distance from the wall y, whether this is determined by the notation ξ_0 or ξ_1. Thus, in both graphs the location of the curves with respect to each other corresponds to the real conditions. Besides the solution ω, for the calculation of which three steps were required, the isothermal velocity profiles $(\omega)_0$ and $(\omega)_1$ for constant properties at the temperatures T_0 and T_1, respectively, were plotted as a function of the non-dimensional coordinates ξ_0 and ξ_1. For the "isothermal" temperature profiles $(\theta)_0$ and $(\theta)_1$, the Pr numbers also were substituted at

the temperatures T_o and T_1. For example, in Figure 2 $Pr_o = 12.5$ and $Pr_1 = 100$. In the following table, τ_o and α denote the shear stress at the wall and heat transfer coefficient $\left[\alpha = \lambda\left(\frac{\partial\theta}{\partial y}\right)\right]$; $(\tau_o)_1$ and $(\alpha)_o$ are the corresponding values at isothermal flow with the viscosity at the temperature T_1 and T_o, respectively. The same holds for $(\tau_o)_o$ and $(\alpha)_1$.

Table 1

	$\frac{\mu_o}{\mu_1}$	Pr_o	$\frac{\tau_o}{(\tau_o)_1}$	$\frac{\tau_o}{(\tau_o)_o}$	$\frac{\alpha}{(\alpha)_o}$	$\frac{\alpha}{(\alpha)_1}$	$\left(\frac{d\omega}{d\xi}\right)_o$	$\left(\frac{d\theta}{d\xi_o}\right)_o$
Heated wall	0.125	12.5	0.841	2.38	1.20	1.70	1.58	1.84
Cooled wall	8,000	100.0	1.08	0.382	0.98	0.694	0.255	3.01

Although in these cases the thickness of the thermal boundary layer is by no means negligible as compared with the flow boundary layer, the shear stress can be calculated very well with the aid of the isothermal formula with the viscosity at the temperature T_1.* The stipulations of Eq. (19) for $Pr > 10$ are thus given.

The conditions for the heat transfer coefficient are more involved. From (8), it follows that the heat transfer coefficient α is proportional to $g(Pr)\sqrt{\frac{U}{\nu x}}$, where, according to E. Polhausen, $g = 0.664\sqrt[3]{Pr}$ is true with high accuracy. Considering that $Pr = \frac{\nu}{a}$, it follows that the heat transfer coefficient is inversely proportional to the sixth root of the viscosity. Thus, the ratio of heat transfer coefficients $(\alpha)_o$ to $(\alpha)_1$ is

$$\frac{(\alpha)_o}{(\alpha)_1} = \left(\frac{\mu_1}{\mu_o}\right)^{\frac{1}{6}}. \qquad (22)$$

A second indication for the heat transfer coefficient is obtained from the velocity gradient at the wall. From Eq. (19), with consideration of (12), it follows (if the indices at the brackets, i_o and i_1, denote isothermal flow at the temperature T_o and T_1, respectively):

* Refer to column $\frac{\tau_o}{(\tau_o)_1}$ in Table 1 and compare it with column $\frac{\tau_o}{(\tau_o)_o}$.

$$\frac{\left(\frac{\partial u}{\partial y}\right)_o}{\left[\left(\frac{\partial u}{\partial y}\right)_o\right]_{i_1}} = \frac{\mu_1}{\mu_o}$$
$$\frac{\left(\frac{\partial u}{\partial y}\right)_o}{\left[\left(\frac{\partial u}{\partial y}\right)_o\right]_{i_o}} = \sqrt{\frac{\mu_1}{\mu_o}} . \qquad (23)$$

These relations also are confirmed by Figures 2 and 3.

Obviously, by means of these formulas, limits can be given for the heat transfer coefficient α (also compare Figs. 2 and 3), which we shall express in terms of multiples of the heat transfer coefficient $(\alpha)_o$. One limit is given by $(\alpha)_1 = \sqrt[6]{\frac{\mu_o}{\mu_1}}\,(\alpha)_o$ (according to Eq. 22), because the velocity profile $(\omega)_1$ yields higher velocities on all points with a cooled plate and lower velocities with a heated plate. The other limit is given by a velocity profile of isothermal form for which the abscissa scale is changed by a factor γ in such a manner that its gradient at the wall coincides with that of the actual velocity distribution. Since the limit mentioned above for the heat transfer coefficient is to be expressed by $(\alpha)_o$ multiplied by a factor, γ is to be calculated from the ratio of the true velocity gradient to that at the temperature T_o (second part of Eq. 23). From the notes on the convergence of the solution method in paragraph II, it then follows that the heat transfer coefficient is proportional to $\sqrt[3]{\gamma}$ and we obtain as the limit $\sqrt[6]{\frac{\mu_1}{\mu_o}}\,(\alpha)_o$. Summarizing, we obtain for the limits of the heat transfer coefficient, if only viscosity varies,

$$\sqrt[6]{\frac{\mu_o}{\mu_1}}\,(\alpha)_o \lesseqgtr \alpha \lesseqgtr \sqrt[6]{\frac{\mu_1}{\mu_o}}\,(\alpha)_o , \qquad (24)$$

where the upper signs stand for the case of the heated plate and the lower signs, for the cooled plate. Accordingly, in agreement with the special examples in Table 1, the heat transfer coefficient $(\alpha)_o$ approaches closest the true heat transfer coefficient α.

Thus, for viscous fluids ($Pr > 10$) for which variation of the viscosity is of primary consideration, the following approximate rule is obtained: In the calculation of the drag, the properties are to be referred to the temperature at the outer limit of the boundary layer; in the calculation of the heat transfer, the properties are to be referred to the wall temperature.

IV. The Flow and Temperature Fields for the Case Pr = 0.7(Air), where All Properties Are Functions of Temperature

In the temperature range -50° to $140^{\circ}C$, the properties of air can be represented by the following formulas:

$$\mu = K_1 T^{0.780}; \quad \rho = K_2 T^{-1}; \quad \lambda = K_3 T^{0.821};$$

where T is temperature in absolute degrees.

With

$$\theta = \frac{T_o - T_1}{T_1}$$

we obtain

$$\frac{\mu}{\mu_1} = \phi = 1 + \theta(1 - \theta)^{0.78}$$

and similar expressions for σ and χ.

The calculation according to the method given in paragraph II was carried out for a heated plate with the values of $\theta = 1/4$ and $1/2$ and, in the velocity and temperature fields, yielded only moderate deviations from the form at isothermal flow (Table 2). In order to calculate the conditions at greater temperature differences, the case of $T_1 = 20^{\circ}C$ and $T_o = 620^{\circ}C$ ($\theta = 2.05$) was calculated. According to Figure 4, the velocity and temperature fields show a substantial variation of the form at constant properties. Again, ξ_o and ξ_1 are formed with the properties at the temperatures T_o and T_1, respectively. For both fields, a substantial increase in the boundary layer thickness is obtained. Nevertheless, the values of the wall shear stress and of the heat transfer coefficient show but little deviation from the values at constant properties, where it is unimportant whether they are referred to the temperature at the wall or to that at the outer limit of the boundary layer. In the

case of air, this can be explained by the fact that, wit in the boundary layer, the increase in viscosity with increasing temperature acts as a drag increase, the decrease in density acts as a drag decrease, and both affects practically eliminate each other at Pr = 0.7, where thermal and flow boundary layers are approximately equal in size. The conditions for the temperature field are almost exactly alike because thermal conductivity is related with temperature, as is viscosity.

Table 2

Heating	$\frac{\tau_o}{(\tau_o)_1}$	$\frac{\tau_o}{(\tau_o)_o}$	$\frac{\alpha}{(\alpha)_1}$	$\frac{\alpha}{(\alpha)_o}$	$\left(\frac{d\theta}{d\xi_1}\right)_o$	$\left(\frac{d\omega}{d\xi_1}\right)_o$
$\theta = 1/4$	1.00	1.02	1.00	1.01	0.486	0.556
$\theta = 1/2$	1.00	1.05	1.00	1.02	0.420	0.485
$T_o = 620°C$ $T_1 = 20°C$	0.93	1.11	0.96	1.03	0.235	0.286

In similar manner, frictional heating can be taken into account, in which case a term $\mu\left(\frac{\partial u}{\partial y}\right)^2$ must be added on the right side of Eq. (11), and the solution becomes

$$\theta = \frac{1 + B(\infty)}{A(\infty)} A(\xi) - B(\xi),$$

$$A(\xi) = \int_0^{\xi} \frac{1}{\chi} e^{-R(\xi)} d\xi,$$

$$B(\xi) = 2\frac{Pr_k \Delta T_e}{(T_1 - T_o)} \int_0^{\xi} \frac{1}{\chi}\left[\int_0^{\xi} \phi\,\omega'^2 e^{-R(\xi)} d\xi\right] e^{-R(\xi)} d\xi, \tag{25}$$

$$R(\xi) = Pr_k \int_0^{\xi} \frac{f(\xi)}{\chi} d\xi,$$

$$\Delta T_e = \frac{U^2}{2c_p}.$$

Here, also, the iteration method can be carried out even though the calculation work involved is somewhat greater. For constant properties, Eq. (25) reduces to the solution given by E. Eckert.[10] By a suitable selection of the constant of integration in (25), even the thermometer problem (vanishing temperature gradient at the wall) can be solved.

Considering Crocco's previous calculations for a gas with $Pr = 0.725$, a numerical example worked out by this new method has been omitted here.

V. Application to a Diffusion Problem

The concentration field in the diffusion problem for the flat plate can be treated in the same manner as the temperature field.*† The differential equation is

$$u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} = k \frac{\partial^2 c}{\partial y^2} , \qquad (26)$$

where k is the diffusion coefficient and c is the concentration that is defined as a quantity of gas or vapor in unit volume. Let us, in this case, consider the properties to be constant, and, in particular, let us assume that the densities of the two interdiffusing media are approximately equal in the entire flow field. On the other hand, let us take into account that, at higher concentrations, the velocity v at the wall no longer vanishes, a fact pointed out previously by Nusselt.[6] When fluid evaporates from a wall, say, because of a gas (e.g., air) flowing past a wetted wall, then some substance enters continuously into the flow; therefore, we obtain at the wall $v(0) > 0$. If, on the other hand, the vapor from a vapor gas mixture condenses on the wall, or if, for example, ammonia containing air flows past a blotting paper saturated with hydrochloric acid, then we

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* E. Eckert reported on a solution of this problem at a meeting of the VDI-Ausschuss für Wärmeforschung, 1943, in Bayreuth, where an approximation method was applied similar to K. Polhausen's method for the flow boundary layer.

† An estimate for the problem at hand has been given by G. Damköhler, Z. Elektrochem, 178 (1942).

obtain $v(0) < 0$. According to Eqs. (100) and (101) by Nusselt,[6] the boundary condition for v is

$$- \frac{k}{c_o} \left(\frac{\partial c}{\partial y}\right)_o \frac{1}{\frac{p}{p_o} - 1} = v(0), \tag{27}$$

where c denotes the concentration of the gas or vapor for which the wall is permeable; c_o, the corresponding concentration at the wall; p_o, the associated partial pressure; and p, the total pressure.

For the velocity profile, the solution of Blasius[2] is now no longer obtained, because $v(0) \neq 0$ (see Fig. 1), but is a family of profiles, depending on the value of $v(0)$. By introducing the free-stream velocity U and the non-dimensional ξ, we obtain

$$\left(\frac{v}{U}\right)_o = -\frac{1}{2} \frac{k}{c_o} \frac{c_1 - c_o}{\left(\frac{p}{p_o} - 1\right) U} \left(\frac{dC}{d\xi}\right)_o \sqrt{\frac{U}{\nu x}} , \quad C = \frac{c - c_o}{c_1 - c_o} , \tag{28}$$

where c_o and c_1 denote the concentrations at the wall and at the outer limit of the boundary layer. Hence, similar to (15),

$$\frac{v}{U} = \sqrt{\frac{\nu}{Ux}} \left\{\omega x - \int_0^\xi \omega d\xi + \frac{M}{2}\right\} , \tag{29}$$

$$M = - \frac{k}{\nu} \frac{c_1 - c_o}{c_o\left(\frac{p}{p_o} - 1\right)} \left(\frac{dC}{d\xi}\right)_o .$$

First, let us state the solution for the concentration field which is, analogously to (18),

$$C = \frac{L(\xi)}{L(\infty)} , \quad L(\xi) = \int_0^\xi e^{- \frac{\nu}{k} \int_0^\xi (f(\xi) - M) d\xi} d\xi \tag{30}$$

$$f(\xi) = 2 \int_0^\xi \frac{u}{U} d\xi ,$$

where $\frac{\nu}{k}$ is a value analogous to the Pr number. In order to obtain the velocity $\omega = \frac{u}{U}$, we simply write in Eq. (30), $\frac{\nu}{k} = 1$. Calculation of

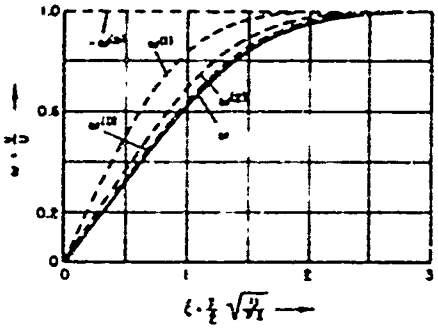
the velocity and concentration fields is accomplished by an iteration method similar to that in paragraph II.

M incorporates the concentration gradient at the wall; however, (30) can be solved for any M values and we can then evaluate $N = -\frac{k}{\nu}\frac{c_1 - c_o}{c_o\left(\frac{p}{p_o} - 1\right)}$ by means of the value for $\left(\frac{dC}{d\xi}\right)_o$ obtained from the solution. The velocity and concentration fields for the evaluated M and N values may be recognized in Figs. 5 and 6, and the concentration gradient at the wall, in Fig. 7. $M > 0$ denotes evaporation from the wall; $M < 0$, condensation or absorption on the plate. For the ratio $\frac{\nu}{k}$, the value 0.6 was selected, which holds, as a good approximation, for the diffusion of water vapor and ammonia in air.[11] From a rigorous point of view, at the higher concentrations stipulated, density and viscosity of the mixture of the two substances are dependent upon the concentration, and the diffusion coefficient, upon the temperature. With the aid of the method described, such cases can also be calculated. If, besides the diffusion, some heat transfer takes place, then the solution for the concentration field can also be applied to the temperature field, as a good approximation. Similarly, the solution for the concentration field presents information on the heat transfer if air at the plate is blown out or sucked off through, for example, a porous wall, with normal velocities at the wall corresponding with Eq. (28).

VI. Summary

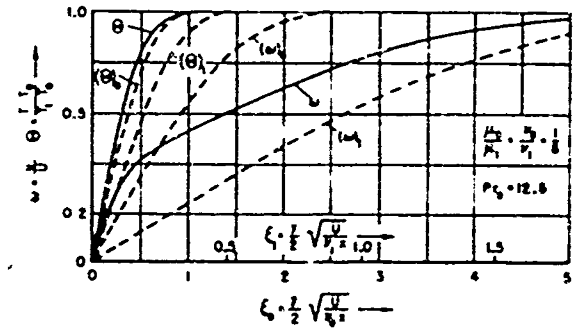
Following closely E. Polhausen's solution for the laminar temperature field at the flat plate in longitudinal flow, formulas are derived which permit calculation of the velocity and temperature field for variable properties by means of an integral equation and an iteration method based on this equation. Accordingly, the following cases were solved: By assuming that only viscosity varies with temperature and that the remaining properties are constant, the velocity and temperature fields were calculated for the Pr numbers 12.5 and 100 (viscous fluids) at heated and cooled plate conditions.

A more rigorous investigation of these two cases yielded an expression of general validity. For a gas with the Pr number 0.7(air), calculations were carried out based on the assumption that all properties vary with temperature and that velocities are not too great, so that frictional heat may be neglected. An increase of the thickness of the boundary layers was obtained without a substantial change, however, of the shear stress or the heat transfer coefficient as compared with those values that were calculated with the formulas for constant properties. In the course of this investigation, it was found that the influences of density and viscosity, and of density and thermal conductivity in the velocity and temperature field, on the wall shear stress and heat transfer coefficient are opposed to each other and that they practically eliminate one another. Formulas which also take into account frictional heating were given, but elaboration of the associated calculations was omitted in consideration of the previous results of the studies by Crocco. Finally, the solution methods developed here were applied also to the case of diffusion of additional substances, where, at higher concentrations, finite normal velocities appear at the wall, which results in a substantial change of the velocity and temperature field.



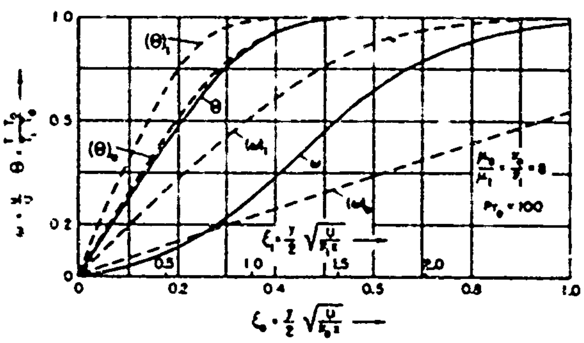
GRAPHICAL REPRESENTATION OF THE INDIVIDUAL APPROXIMATIONS IN THE CALCULATION OF THE VELOCITY DISTRIBUTION ON THE FLAT PLATE ACCORDING TO THE METHOD PRESTON AND PIERCY (CONSTANT PROPERTIES)

FIG. 1



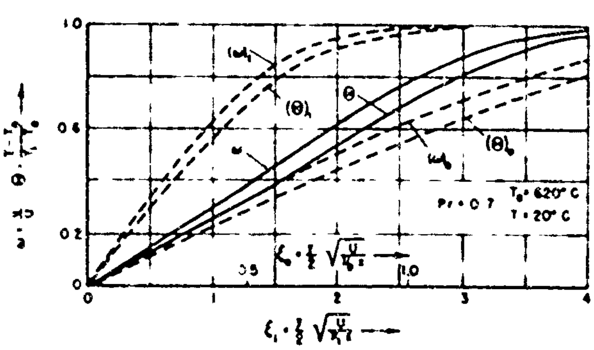
VELOCITY AND TEMPERATURE DISTRIBUTION ON A HEATED PLATE AT VARIABLE VISCOSITY. VISCOSITY EXPONENT $b=3$. $(\omega)_0$, $(\Theta)_0$ AND $(\omega)_1$, $(\Theta)_1$ ARE THE "ISOTHERMAL" VELOCITY AND TEMPERATURE DISTRIBUTION; ν_0 AND ν_1 DENOTE KINEMATIC VISCOSITY AT THE WALL TEMPERATURE T_0 AND AT THE TEMPERATURE T_1 AT THE OUTER LIMIT OF THE BOUNDARY LAYER

FIG. 2



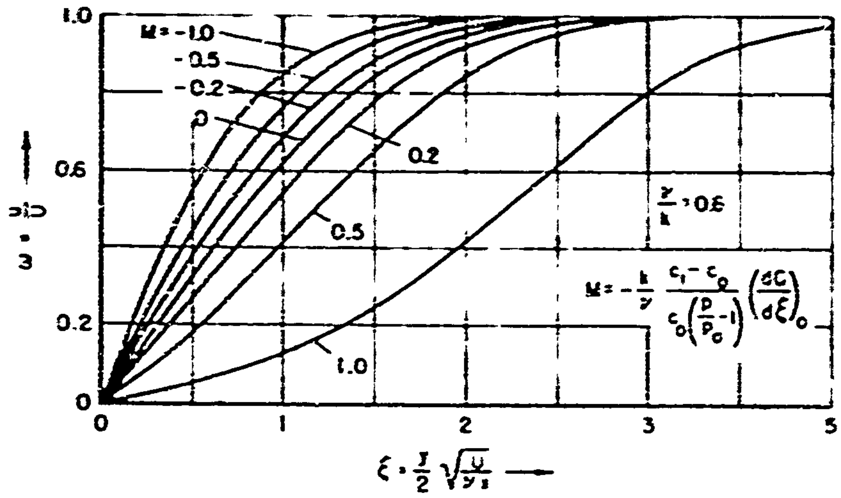
VELOCITY AND TEMPERATURE DISTRIBUTIONS ON THE COOLED PLATE NOTATIONS ARE THE SAME AS IN FIG. 2

FIG. 3



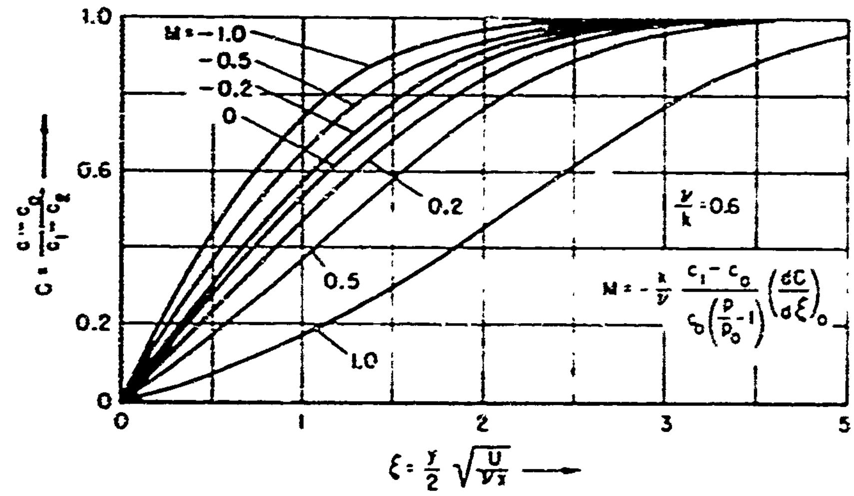
VELOCITY AND TEMPERATURE DISTRIBUTION ON A HEATED PLATE FOR $Pr=0.7$ (AIR); ALL PROPERTIES ARE VARIABLE WITH TEMPERATURE

FIG. 4



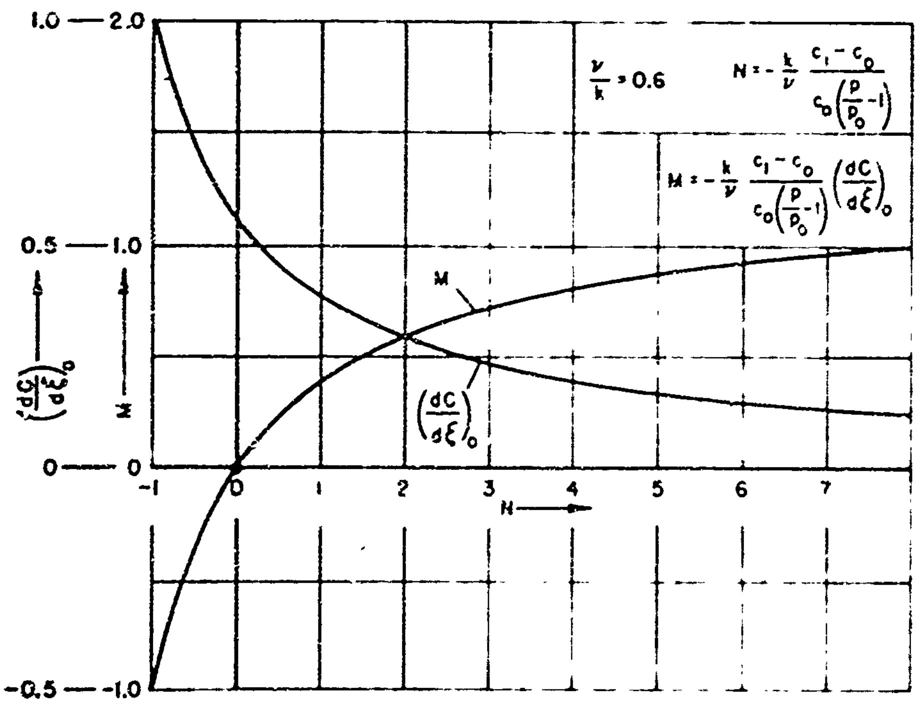
VELOCITY FIELD FOR DIFFUSION AT HIGHER CONCENTRATIONS IN WHICH FINITE NORMAL VELOCITIES APPEAR ON THE WALL (COMPARE TEXT TO EQS. 26 TO 28)

FIG. 5



CONCENTRATION DISTRIBUTION FOR FIG. 5

FIG. 6



CONCENTRATION GRADIENT AT THE WALL AND THE VALUE M AS A FUNCTION OF N (SUPPLEMENT TO FIG. 6)

FIG. 7

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ABSTRACT:

Formulas are derived for calculation of the laminar flow velocity, temperature field and diffusion along a flat plate by means of an integral equation and an iteration method based on this equation. The velocity and temperature fields were calculated for Prandtl numbers 12.5 and 100 at heated and cooled plate conditions, and for a gas with Prandtl number 0.7. An increase of the thickness of the boundary layer was obtained without substantial change of the shear stress or the heat transfer coefficient. Formulas which also take into account frictional heating are given. The methods of solution developed were applied in the case of diffusions of additional substances.

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